

UNIQUENESS OF VISCOSITY SOLUTIONS OF A GEOMETRIC FULLY NONLINEAR PARABOLIC EQUATION

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Abstract. We observe that the comparison result of Barles-Biton-Ley for viscosity solutions of a class of nonlinear parabolic equations can be applied to a geometric fully nonlinear parabolic equation which arises from the graphic solutions for the Lagrangian mean curvature flow.

1. INTRODUCTION

We consider the question of uniqueness for the following fully nonlinear parabolic equation

$$(1.1) \quad \frac{\partial u}{\partial t} = \sum_{j=1}^n \arctan \lambda_j$$

with initial condition $u(x, 0) = u_0(x)$, where u is a function from \mathbb{R}^n to \mathbb{R} and λ_j 's are the eigenvalues of the Hessian D^2u . This equation arises naturally from geometry. In fact, when u is a regular solution to (1.1), it is known that the graph $(x, Du(x, t))$ evolves by the mean curvature flow and it is a Lagrangian submanifold in $\mathbb{R}^n \times \mathbb{R}^n$ with the standard symplectic structure, for each t . Recently, smooth longtime entire solution to (1.1) has been constructed in [2] assuming a certain bound on the Lipschitz norm of Du_0 .

Barles, Biton and Ley have obtained a very useful general comparison result (Theorem 2.1 in [1]) for the viscosity solutions for a class of fully nonlinear parabolic equations, as well as existence result (Theorem 3.1 in [1]). In particular, they showed that (1.1) admits a unique longtime continuous viscosity solution for any continuous function u_0 in \mathbb{R} when $n = 1$.

In this short note, we observe, via elementary methods, that the hypotheses in the general theorems in [1] are valid for the geometric evolution equation (1.1) in general dimensions. The result is the following

Theorem 1.1. *Let u and v be an upper semicontinuous and a lower semicontinuous viscosity subsolution and supersolution to (1.1) in $\mathbb{R}^n \times [0, T)$. If $u(x, 0) \leq v(x, 0)$ for all $x \in \mathbb{R}^n$, then $u \leq v$ in $\mathbb{R}^n \times [0, T)$. In particular, for any continuous function u_0 in \mathbb{R}^n , there is a unique continuous viscosity solution to (1.1) in $\mathbb{R}^n \times [0, \infty)$.*

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2. HYPOTHESES (H1) AND (H2)

We now describe the assumptions in the comparison and existence results in [1].

Let S_n be the linear space of real $n \times n$ symmetric matrices. If $X \in S_n$, there exists an orthogonal matrix P such that $X = P\Lambda P^T$ where Λ is the diagonal matrix with diagonal entries consist of eigenvalues of X . Let Λ^+ be the diagonal matrix obtained by replacing the negative eigenvalues in Λ with 0's. Define $X^+ = P\Lambda^+P^T$.

Consider a continuous function F from $\mathbb{R}^n \times [0, T] \times \mathbb{R}^n \times S_n$ to \mathbb{R} . The following assumptions of F are necessary to apply the results in [1]:

(H1) For any $R > 0$, there exists a fuction $m_R : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $m_R(0^+) = 0$ and

$$F(y, t, \eta(x - y), Y) - F(x, t, \eta(x - y), X) \leq m_R(\eta|x - y|^2 + |x - y|)$$

for all $x, y \in \overline{B}(0, R)$ and $t \in [0, T]$, whenever $X, Y \in S_n$ and $\eta > 0$ satisfy

$$-3\eta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\eta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

(H2) There exists $0 < \alpha < 1$ and constants $K_1 > 0$ and $K_2 > 0$ such that

$$F(x, t, p, X) - F(x, t, q, Y) \leq K_1 |p - q| (1 + |x|) + K_2 (\text{tr}(Y - X)^+)^{\alpha}$$

for every $(x, t, p, q, X, Y) \in \mathbb{R}^n \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times S_n \times S_n$.

The operator F is degenerate elliptic if (H2) holds.

Theorem 2.1. (*Barles-Biton-Lay*) *Let u and V be an upper semicontinuous viscosity subsolution and a lower semicontinuous viscosity supersolution respectively of*

$$\begin{aligned} \frac{\partial u}{\partial t} + F(x, t, Du, D^2u) &= 0 \quad \text{in } \mathbb{R}^n \times [0, T] \\ u(\cdot, 0) &= u_0 \quad \text{in } \mathbb{R}^n. \end{aligned}$$

Assume that (H1) and (H2) hold for F . Then

- (1) *If $u(\cdot, 0) \leq v(\cdot, 0)$ in \mathbb{R}^n , then $u \leq v$ in $\mathbb{R}^n \times [0, T]$.*
- (2) *If $u_0 \in C(\mathbb{R}^n)$ there is a unique continuous viscosity solution in $\mathbb{R}^n \times [0, \infty)$.*

We now present the proof of Theorem 1.1.

Proof. We define $F : S_n \rightarrow \mathbb{R}$ by

$$(2.1) \quad F(X) = -i \log \frac{\det(I + iX)}{\det(I + X^2)^{\frac{1}{2}}} = -\frac{i}{2} \log \frac{\det(I + iX)}{\det(I - iX)}.$$

That F takes real values follows easily from

$$\overline{F(X)} = \frac{i}{2} \log \frac{\det(I - iX)}{\det(I + iX)} = F(X).$$

Note that $F(D^2u)$, by diagonalizing D^2u at a point, is equal to $\sum \arctan \lambda_j$. Therefore the flow (1.1) can be written as

$$u_t + (-F(D^2u)) = 0.$$

Since $F(x, t, p, X) = F(X)$ is independent of x , the right hand side of the inequality for F in (H1) must be zero, namely $m_R = 0$. By multiplying an arbitrary vector $(\xi, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and its transpose to the second matrix inequality in (H1), we see that $X \leq Y$. Therefore, in order to establish (H1) it suffices to show:

(H1') For any $X, Y \in S_n$, if $X \geq Y$ then $F(X) \geq F(Y)$.

For any $X, Y \in S_n$ and $t \in [0, 1]$, define

$$f_{XY}(t) = F(tX + (1 - t)Y).$$

We will show that $f_{XY}(t)$ is nondecreasing in $t \in [0, 1]$ and then (H1') will follow as $f_{XY}(0) = F(Y)$ and $f_{XY}(1) = F(X)$. Set

$$A = I + i(tX + (1 - t)Y)$$

and

$$B = I - i(tX + (1 - t)Y).$$

Then

$$f_{XY}(t) = -\frac{i}{2}(\log \det A - \log \det B).$$

It follows that $AB = BA$ and

$$(A^{-1} + B^{-1}) \cdot \frac{AB}{2} = \frac{A + B}{2} = I.$$

Note that both A and B are invertible matrices for all $t \in [0, 1]$. Hence, by using the formula $\partial_t \ln \det G = \text{tr}(G^{-1} \partial_t G)$ for $G(t) \in GL(n, \mathbb{R})$, we have

$$\begin{aligned} f'_{XY}(t) &= -\frac{i}{2} \text{tr} (A^{-1} \cdot \partial_t A - B^{-1} \cdot \partial_t B) \\ &= -\frac{i}{2} \text{tr} ((A^{-1} + B^{-1}) \cdot i(X - Y)) \\ (2.2) \quad &= \text{tr} ((I + (tX + (1 - t)Y)^2)^{-1} \cdot (X - Y)). \end{aligned}$$

Since $tX + (1 - t)Y$ is real symmetric, the matrix

$$C = I + (tX + (1 - t)Y)^2$$

is positive definite, hence so is C^{-1} . There exists a matrix $Q \in GL(n, \mathbb{R})$ such that $C = QQ^T$. By the assumption $X \geq Y$, we have

$$\begin{aligned} \text{tr} (C^{-1}(X - Y)) &= \text{tr} (Q \cdot Q^T(X - Y)) \\ &= \text{tr} (Q^T(X - Y) \cdot Q) \\ &\geq 0 \end{aligned}$$

since $Q^T(X - Y)Q$ is positive semidefinite. Therefore, we have shown that (H1) is valid for F defined in (2.1).

As $F(x, t, p, X)$ is independent of p , (H2) reads: there exist constants $K > 0$ and $0 < \alpha < 1$ such that $F(X) - F(Y) \leq K (\text{tr}(X - Y)^+)^{\alpha}$ for all $X, Y \in S_n$.

For any $X, Y \in S_n$, integrating (2.2) leads to

$$(2.3) \quad F(X) - F(Y) = \int_0^1 \operatorname{tr}(C^{-1}X) dt.$$

For $X - Y \in S_n$ there exists an orthogonal matrix P such that $X - Y = P\Lambda P^T$ where the diagonal matrix Λ has diagonal entries $\lambda_1, \dots, \lambda_n$. Let $\lambda_j^+ = \max\{\lambda_j, 0\}$. Since $0 < C^{-1} \leq I$, we have $0 < P^T C^{-1} P \leq I$. If c_{jj} denote the diagonal entries of $P^T C^{-1} P$ for $j = 1, \dots, n$, then $c_{jj} = \langle P^T C^{-1} P e_j, e_j \rangle$ where $\{e_1, \dots, e_n\}$ is the standard basis for \mathbb{R}^n and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. It follows that $0 < c_{jj} \leq 1$ for $j = 1, \dots, n$. Then

$$\begin{aligned} \operatorname{tr}(C^{-1}(X - Y)) &= \operatorname{tr}(P^T C^{-1} P \cdot P^T(X - Y)P) \\ &= \operatorname{tr}(P^T C^{-1} P \cdot \Lambda) \\ &= \sum c_{jj} \lambda_j \\ &\leq \sum \lambda_j^+ \\ &= \operatorname{tr}(X - Y)^+. \end{aligned}$$

Substituting the above inequality into (2.3) implies: for any $X, Y \in S_n$ we have

$$F(X) - F(Y) \leq \operatorname{tr}(X - Y)^+.$$

Because $\arctan x$ is in $(-\pi/2, \pi/2)$, we have $F(X) - F(Y) < n\pi$. For any constant α with $0 < \alpha < 1$, if $\operatorname{tr}(X - Y)^+ \leq 1$ then

$$F(X) - F(Y) \leq \operatorname{tr}(X - Y)^+ \leq n\pi \operatorname{tr}[(X - Y)^+]^\alpha$$

and if $\operatorname{tr}(X - Y)^+ > 1$ then

$$F(X) - F(Y) \leq n\pi \leq n\pi \operatorname{tr}[(X - Y)^+]^\alpha.$$

Therefore, (H2) holds for $K_2 = n\pi$ and any constants $K_1 > 0$ and α with $0 < \alpha < 1$.

Now Theorem 1.1 follows immediately from Theorem 2.1. \square

We remark on that (H1'), for the operator $F(X) = \sum \arctan \lambda_j(X)$, also follows from the basic fact (cf. p.182 in [4]): Suppose that $X, Y \in S_n$ and the eigenvalues λ_j 's of X and μ_j 's of Y are in descending order $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$. If $X \geq Y$, then $\lambda_j \geq \mu_j$ for $j = 1, \dots, n$.

We also mention the uniqueness of viscosity solutions of the Cauchy-Dirichlet problem for (1.1). Note that the operator $F(X) = \sum \arctan \lambda_j(X)$ satisfies (H1') which is exactly the fundamental monotonicity condition (0.1) for $-F$ in [3], therefore $-F$ is proper in the sense of [3] (cf. p.2 in [3]). As (H1) holds, Theorem 8.2 in [3] is valid for (1.1):

Theorem 2.2. *The continuous viscosity solution to the following Cauchy-Dirichlet problem is unique:*

$$\begin{aligned} u_t &= \sum_{j=1}^n \arctan \lambda_j, \quad \text{in } (0, T) \times \Omega \\ u(t, x) &= 0, \quad \text{for } 0 \leq t < T \text{ and } x \in \partial\Omega \\ u(0, x) &= \psi(x), \quad \text{for } x \in \overline{\Omega} \end{aligned}$$

where λ_j 's are the eigenvalues of D^2u , $\Omega \subset \mathbb{R}^n$ is open and bounded and $T > 0$ and $\psi \in C(\overline{\Omega})$. If u is an upper semicontinuous viscosity solution and v is a lower semicontinuous viscosity solution of the Cauchy-Dirichlet problem, then $u \leq v$ on $[0, T) \times \Omega$.

Note that the initial-boundary conditions for the subsolution and supersolution are: $u(x, t) \leq 0 \leq v(x, t)$ for $t \in [0, T)$ and $x \in \partial\Omega$ and $u(x, 0) \leq \psi(x) \leq v(x, 0)$ for $x \in \overline{\Omega}$.

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